Transitivity of Bender–Knuth Moves on Standard and Semistandard Young Tableaux

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Proof School

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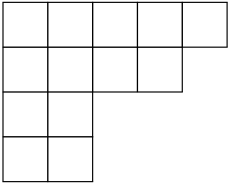
Standard Young Tableaux



Standard Young Tableaux

1	3	5	9	10
2	4	7	11	
6	8			
12	13			

Bender–Knuth Moves

The operation t_i swaps the numbers i and $i+1$ if possible.

1	3	5	9	10
2	4	7	11	
6	8			
12	13			

Bender–Knuth Moves

The operation t_i swaps the numbers i and $i+1$ if possible.

1	3	5	9	10
2	4	7	11	
6	8			
12	13			

Applying t_2...

Bender–Knuth Moves

The operation t_i swaps the numbers i and $i+1$ if possible.

1	2	5	9	10
3	4	7	11	
6	8			
12	13			

Applying t_2...

Bender–Knuth Moves

The operation t_i swaps the numbers i and $i+1$ if possible.

1	2	5	9	10
3	4	7	11	
6	8			
12	13			

Applying t_6...

Bender–Knuth Moves

The operation t_i swaps the numbers i and $i+1$ if possible.

1	2	5	9	10
3	4	6	11	
7	8			
12	13			

Applying t_6...

Bender–Knuth Moves

The operation t_i swaps the numbers i and $i+1$ if possible.

1	2	5	9	10
3	4	6	11	
7	8			
12	13			

Applying t_9...

The operation t_i swaps the numbers i and $i + 1$ if possible.

1	2	5	9	10
3	4	6	11	
7	8			
12	13			

Applying t_9...

...Does not result in a valid standard Young tableaux.

Is there a sequence of t_i moves between these standard Young tableaux?

1	2	5	6
3	4	8	
7			

1	2	3	4
5	6	7	
8			

Transitivity

Is there a sequence of t_i moves between these standard Young tableaux?

1	2	4	6
3	5	8	
7			

1	2	3	4
5	6	7	
8			

Transitivity

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8			

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8			

Transitivity

Is there a sequence of t_i moves between these standard Young tableaux?

1	2	3	4
5	6	7	
8			

1	2	3	4
5	6	7	
8			

Transitivity

Is there a sequence of t_i moves between these standard Young tableaux?

1	2	3	4
5	6	7	
8			

1	2	3	4
5	6	7	
8			

Yes! But is this always the case?

Double Transitivity?

1	2	5	6
3	4		

1	2	4	5
3	6		

Double Transitivity?

1	2	5	6
3	4		

1	2	4	5
3	6		

$\downarrow t_3$

1	2	5	6
3	4		

1	2	3	5
4	6		

Double Transitivity?

1	2	5	6
3	4		

1	2	4	5
3	6		

$\downarrow t_3$

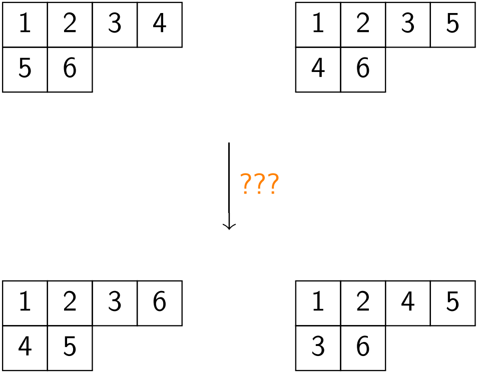
1	2	5	6
3	4		

1	2	3	5
4	6		

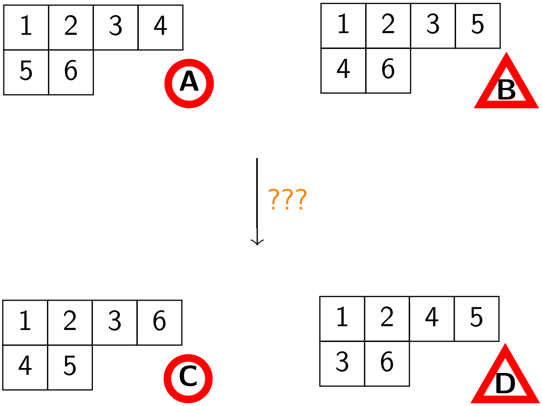
$\downarrow t_4$

1	2	4	6
3	5		

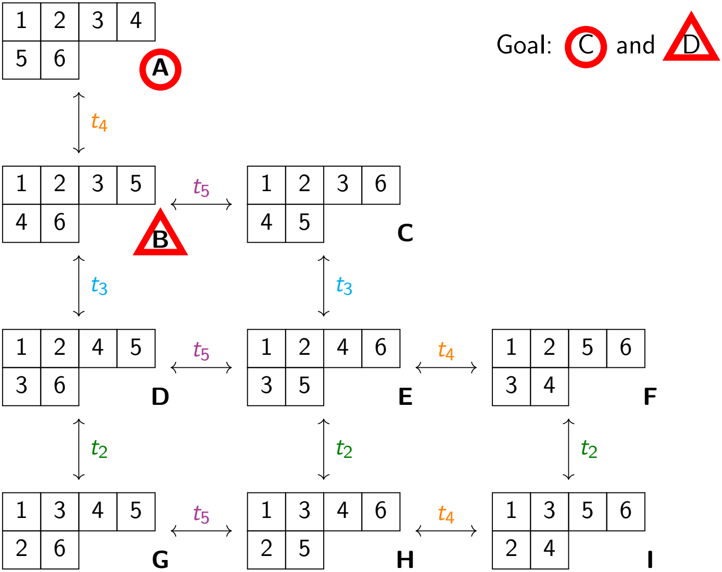
1	2	3	4
5	6		


1 2 3 4
5 6
1 2 3 5
4 6
???
1 2 3 6
4 5
1 2 4 5
3 6

Double Transitivity?



Double Transitivity?

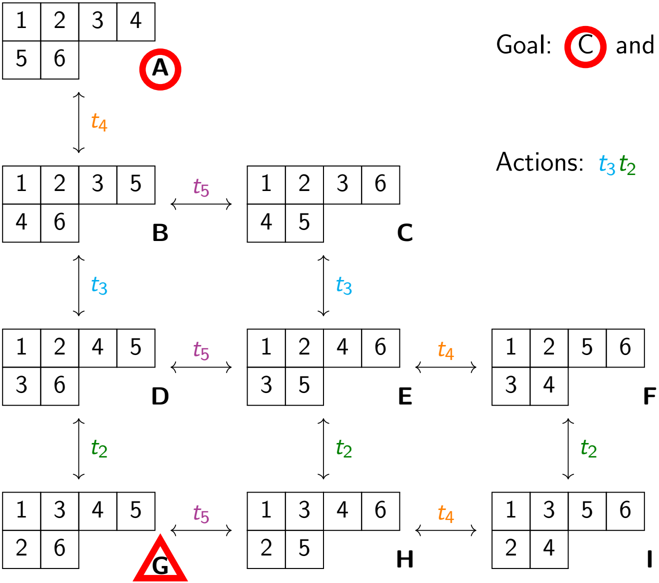


Double Transitivity?

Goal: C and D

Actions: t_3

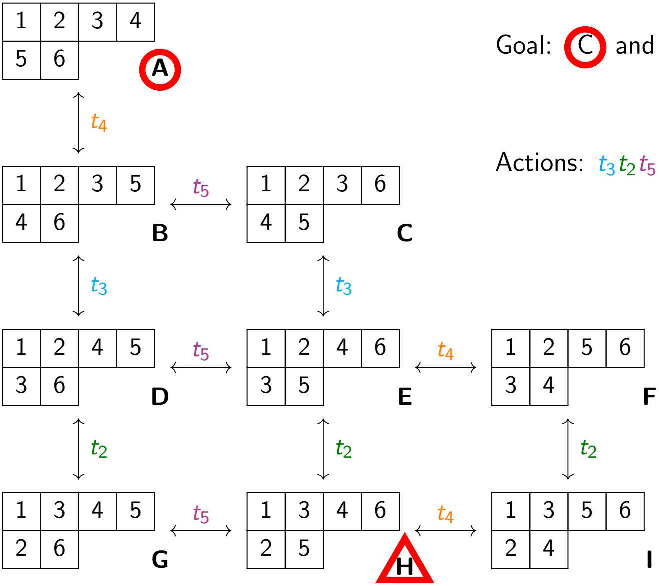
Double Transitivity?



Goal: C and D

Actions: $t_3 t_2$

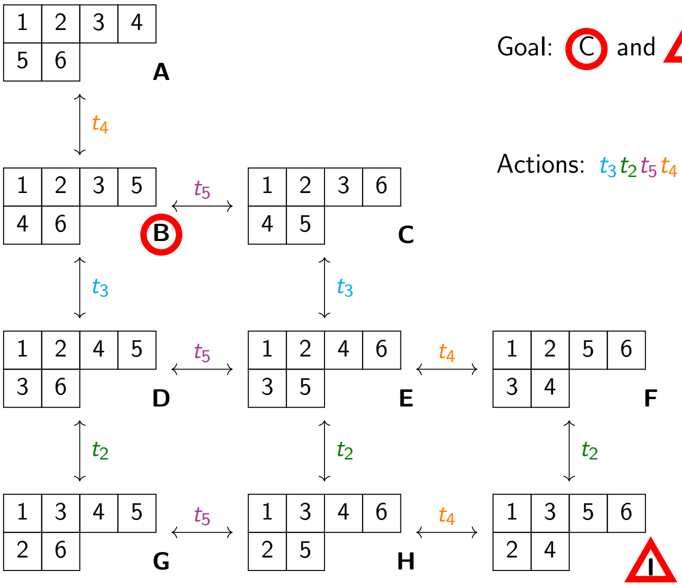
Double Transitivity?



Goal: C and D

Actions: $t_3 t_2 t_5$

Double Transitivity?



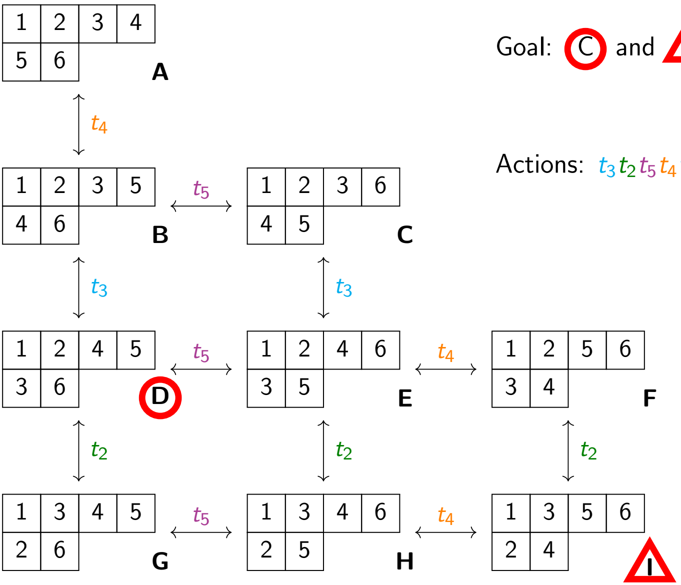
Goal: C and D

Actions: $t_3 t_2 t_5 t_4$

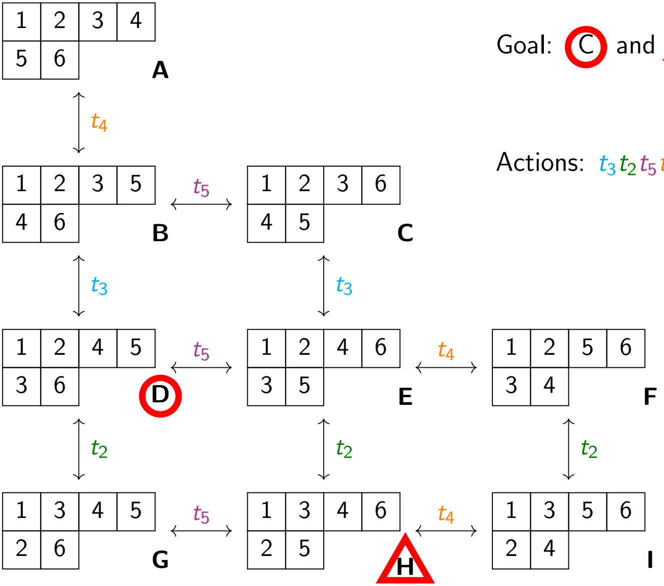
Double Transitivity?

Goal: C and D

Actions: $t_3 t_2 t_5 t_4 t_3$



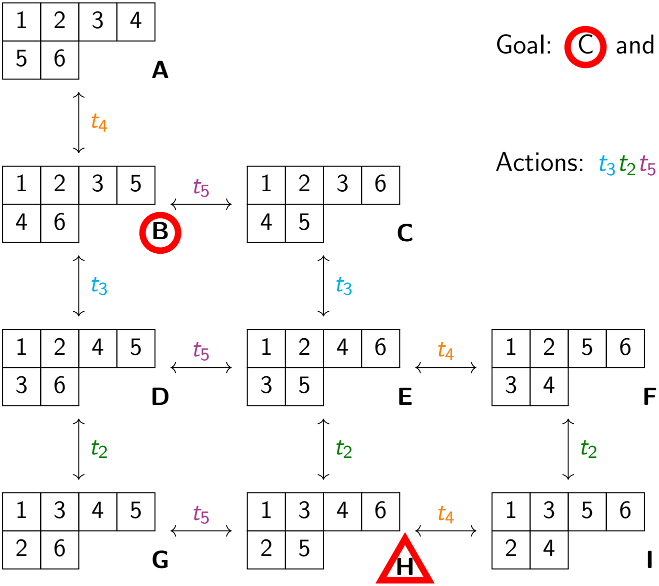
Double Transitivity?



Goal: C and D

Actions: $t_3 t_2 t_5 t_4 t_3 t_4$

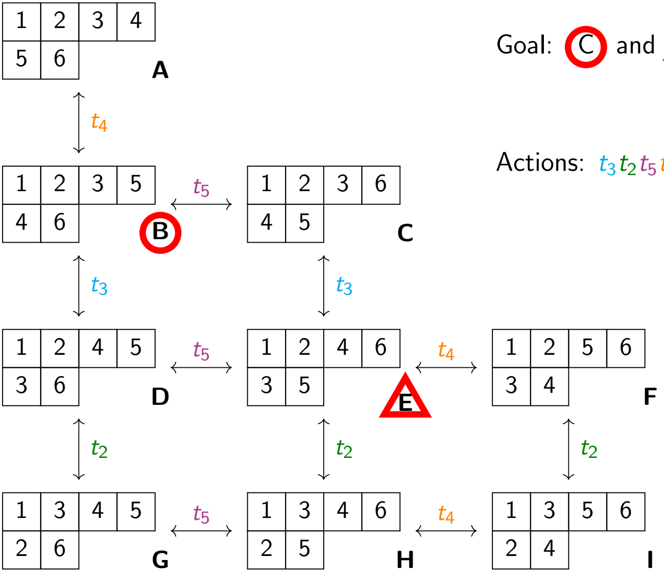
Double Transitivity?



Goal: C and D

Actions: $t_3\, t_2\, t_5\, t_4\, t_3\, t_4\, t_3$

Double Transitivity?



Goal: C and D

Actions: $t_3\, t_2\, t_5\, t_4\, t_3\, t_4\, t_3\, t_2$

Double Transitivity?

Goal: C and D

Actions: $t_3\, t_2\, t_5\, t_4\, t_3\, t_4\, t_3\, t_2\, t_5$

Double Transitivity!

Theorem (Liao 2024)

Let λ be a partition which is not hook-shaped. If $\lambda^T \neq \lambda$, then there are 2 orbits for pairs of standard Young tableaux of λ under the Bender–Knuth moves.

Double Transitivity!

Theorem (Liao 2024)

Let λ be a partition which is not hook-shaped. If $\lambda^T \neq \lambda$, then there are 2 orbits for pairs of standard Young tableaux of λ under the Bender–Knuth moves.

Theorem (Liao 2024)

Let λ be a partition which is not hook-shaped. If $\lambda^T = \lambda$, then there are 3 orbits for pairs of standard Young tableaux of λ under the Bender–Knuth moves.

Conjecture (Liao 2024)

Let λ be a partition which is not hook-shaped. If $\lambda^T \neq \lambda$, then there are 5 orbits for triples of standard Young tableaux of λ under the Bender–Knuth moves.

n-Transitivity

Conjecture (Liao 2024)

Let λ *be a partition which is not hook-shaped. If* $\lambda^T \neq \lambda$, *then there are* **"as few orbits as possible"** *for n-tuples of standard Young tableaux of* λ *under the Bender–Knuth moves.*

n-Transitivity

Conjecture (Liao 2024)

Let λ be a partition which is not hook-shaped. If $\lambda^T \neq \lambda$, then there are **"as few orbits as possible"** *for n-tuples of standard Young tableaux of λ under the Bender–Knuth moves.*

Conjecture (Liao 2024)

Let λ be a partition which is not hook-shaped. If $\lambda^T \neq \lambda$ and $\mathrm{SYT}(\lambda) \geq n$*, then there are*

$$\sum_{\ell=1}^{k} \left(\frac{2^{k-\ell}}{(\ell-1)!} \cdot \sum_{i=0}^{\ell-1} (-1)^i \binom{\ell-1}{i} (\ell-i)^{k-1} \right)$$

orbits for n-tuples of standard Young tableaux of λ under the Bender–Knuth moves.

Semistandard Young Tableaux

The columns strictly increase, but the rows are nondecreasing.

1	1	1	2	2	2	3	3	3
2	2	3	3	3				
4	4	5						
5	6							

Semistandard Young Tableaux

The operation t_i swaps the numbers i and $i+1$ if possible.

1	1	1	2	2	2	3	3	3
2	2	3	3	3				
4	4	5						
5	6							

Semistandard Young Tableaux

The operation t_i swaps the numbers i and $i+1$ if possible.

1	1	1	2	2	2	3	3	3
2	2	3	3	3				
4	4	5						
5	6							

Applying t_2...?

Semistandard Young Tableaux

The operation t_i ~~swaps the numbers i and $i+1$ if possible.~~

1	1	1	2	2	2	3	3	3
2	2	3	3	3				
4	4	5						
5	6							

Applying t_2...?

Semistandard Young Tableaux

The operation t_i swaps the **number of occurrences** of i and $i+1$ in each row.

1	1	1	2	2	2	3	3	3
2	2	3	3	3				
4	4	5						
5	6							

Applying t_2...?

Semistandard Young Tableaux

The operation t_i swaps the **number of occurrences** of i and $i+1$ in each row.

1	1	1	2	2	2	3	3	3
2	2	2	3	3				
4	4	5						
5	6							

Applying t_2...?

Semistandard Young Tableaux

The operation t_i swaps the **number of occurrences** of i and $i+1$ in each row.

1	1	1	2	2	2	3	3	3
2	2	2	3	3				
4	4	5						
5	6							

Applying t_2...?

But what if we started with a different tableaux?

Semistandard Young Tableaux

The operation t_i swaps the **number of occurrences** of i and $i+1$ in each row.

1	1	1	2	2	2	3	3	3
2	3	3	3	3				
4	4	5						
5	6							

Applying t_2...?

Semistandard Young Tableaux

The operation t_i swaps the **number of occurrences** of i and $i+1$ in each row.

1	1	1	2	2	2	3	3	3
2	2	2	2	3				
4	4	5						
5	6							

Applying t_2...?

The operation t_i ~~swaps the number of occurrences of i and $i + 1$ in each row.~~

1	1	1	2	2	2	3	3	3
2	2	2	2	3				
4	4	5						
5	6							

Applying t_2...?

Semistandard Young Tableaux

The operation t_i ~~swaps the number of occurrences of i and $i+1$ in each row.~~

1	1	1	2	2	2	3	3	3
2	3	3	3	3				
4	4	5						
5	6							

Applying t_2...?

Semistandard Young Tableaux

The operation t_i ~~swaps the number of occurrences of i and $i+1$ in each row.~~

1	1	1	2	2	2	3	3	3
2	3	3	3	3				
4	4	5						
5	6							

Applying t_2...?

Semistandard Young Tableaux

The operation t_i swaps the number of occurrences of i and $i+1$ in each row, excluding "locked columns."

1	1	1	2	2	2	3	3	3
2	3	3	3	3				
4	4	5						
5	6							

Applying t_2...

Semistandard Young Tableaux

The operation t_i swaps the number of occurrences of i and $i+1$ in each row, excluding "locked columns."

1	1	1	2	2	2	2	2	3
2	2	3	3	3				
4	4	5						
5	6							

Applying t_2...

The operation t_i swaps the number of occurrences of i and $i+1$ in each row, excluding "locked columns."

1	1	1	2	2	2	2	2	3
2	2	3	3	3				
4	4	5						
5	6							

Applying t_1...

Semistandard Young Tableaux

The operation t_i swaps the number of occurrences of i and $i+1$ in each row, excluding "locked columns."

1	1	1	2	2	2	2	2	3
2	2	3	3	3				
4	4	5						
5	6							

Applying t_1...

The operation t_i swaps the number of occurrences of i and $i+1$ in each row, excluding "locked columns."

1	1	1	2	2	2	2	2	3
2	2	3	3	3				
4	4	5						
5	6							

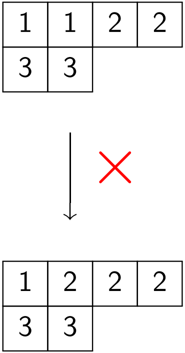
Applying t_1...

The operation t_i swaps the number of occurrences of i and $i+1$ in each row, excluding "locked columns."

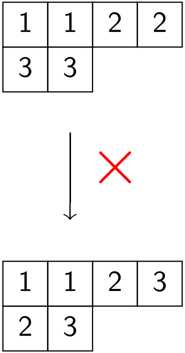
1	1	1	1	1	1	1	2	3
2	2	3	3	3				
4	4	5						
5	6							

Applying t_1...

Semistandard Young Tableaux Are Not Transitive



Semistandard Young Tableaux Are Not Transitive (at all)



But They Are Sometimes a Little Transitive

Theorem (Liao 2024)

Let λ be a 2-row partition. Then there is a sequence of Bender–Knuth moves between two semistandard Young tableaux of λ if their set of counts are the same and includes a count of 1.

But They Are Sometimes a Little Transitive

Theorem (Liao 2024)

Let λ be a 2-row partition. Then there is a sequence of Bender–Knuth moves between two semistandard Young tableaux of λ if their set of counts are the same and includes a count of 1.

1	2	2	2
3	3		

↓ ✓

1	2	2	3
2	3		

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- MIT PRIMES

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